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NOTE

A CHARACTERIZATION OF COMPETITION GRAPHS OF ARBITRARY DIGRAPHS*

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Let $D=(V,A)$ be a digraph which may have loops, i.e., arcs of the form (x,x). The *competition graph* of D is the undirected graph G obtained as follows. $V(G)=V$ and $\{x,y\}\in E(G)$ if and only if for some $z\in V$, arcs (x,z) and (y,z) are in A.

Competition graphs were introduced by Cohen [1968] in connection with problems of ecology. Cohen was primarily interested in the case where D is an acyclic digraph. This case is also discussed by Cohen [1977, 1978], Dutton and Brigham [1983], Lundgren and Maybee [1983], Opsut [1982], and Roberts [1978a,b]. Dutton and Brigham [1983] introduce the study of competition graphs of digraphs D which may not be acyclic. A notion similar to that of competition graph arises when D is a bipartite digraph with all arcs heading from a set S to a set T. The subgraph of the competition graph which is generated by vertices in S is the confusion graph which arises in the study of communication over noisy channels (Shannon [1956], Roberts [1978b]). Here, the set S is a transmission alphabet and the set T is a receiving alphabet, and an arc from s to t means that s can be received as t. A similar construction gives rise to the conflict graph which is relevant to the problem of assigning channels to radio or television transmitters (Cozzens and Roberts [1982], Hale [1980], Opsut and Roberts [1981]). Here, the set S is taken to be a set of transmitters and the set T to be a set of locations, and an arc from s to t means that transmitter s can be heard at location t.

Dutton and Brigham characterize the competition graphs arising from digraphs D which may have cycles and which also may have loops. We shall obtain a similar characterization in the case that there are no loops.

Let $m(G)$ be the smallest number of cliques (not necessarily maximal) of G which cover all edges of G. Orlin [1977] and Kou, et al. [1978] prove that computation of $m(G)$ is an NP-complete problem. Dutton and Brigham [1983] prove the following theorem.

Theorem 1 (Dutton and Brigham). *If $|V(G)|=n$, then G is a competition graph of a digraph D (which may have loops) if and only if $m(G)\le n$.*

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We shall prove:

Theorem 2. *If $|V(G)|=n$, then G is a competition graph of a digraph which has no loops if and only if $G \neq K_2$ and $m(G) \leq n$.*

To prove Theorem 2, we first observe:

Theorem 3. *G is a competition graph of a digraph which has no loops if and only if there are cliques $C_1, C_2, \ldots, C_p$ which cover the edges of G and such that if $D_i = V(G) - C_i$, then $\{D_1, D_2, \ldots, D_p\}$ has a system of distinct representatives.*

Proof. Suppose $\{D_1, D_2, \ldots, D_p\}$ has a system of distinct representatives $(v_1, v_2, \ldots, v_p)$. Let D be a digraph on $V = V(G)$ with an arc from every vertex of C_i to v_i. Then note that $v_i \notin C_i$ implies D has no loops. Moreover, G is the competition graph of D.

Conversely, suppose G is the competition graph of $D = (V, A)$. Let $v_1, v_2, \ldots, v_n$ be the vertices of D, and let $C_i = \{v : (v, v_i) \in A\}$. Then each C_i is a clique and each edge of G is covered by some C_i. Moreover, $v_i \notin C_i$ since D has no loops, and so $(v_1, v_2, \ldots, v_n)$ is a system of distinct representatives for $\{D_1, D_2, \ldots, D_n\}$. $\square$

Definition. A *k-formation* is a collection $T_1, T_2, \ldots, T_k$ of subsets of $\{1, 2, \ldots, k-1\}$ such that no T_i is contained in $\bigcup_{j \neq i} T_j$.

Lemma. *For $k \geq 2$, there is no k-formation.*

Proof. The proof is by induction on k. If $k=2$, the only subsets of $\{1, 2, \ldots, k-1\}$ are $\emptyset$ and $\{1\}$ and clearly there is no k-formation. Suppose the lemma is true for $k-1$ and suppose $T_1, T_2, \ldots, T_k$ is a k-formation. We reach a contradiction. Since $T_1, T_2, \ldots, T_k$ is a k-formation, at least one element of T_k is not contained in any other T_i. Suppose without loss of generality this element is $k-1$. Let $T_i' = T_i - \{k-1\}$. Then $T_1', T_2', \ldots, T_{k-1}'$ is a $(k-1)$-formation. For each T_i' is a subset of $\{1, 2, \ldots, k-2\}$. Moreover, suppose

$$T_i' \subseteqq \bigcup_{\substack{j=1 \\ j \neq i}}^{k-1} T_j'.$$

Then

$$T_i \subseteqq T_i' \cup \{k-1\} \subseteqq \bigcup_{\substack{j=1 \\ j \neq i}}^{k-1} T_j' \cup \{k-1\} \subseteqq \bigcup_{\substack{j=1 \\ j \neq i}}^{k} T_j$$

where the latter containment follows since $k-1 \in T_k$. Thus, we have a $(k-1)$-formation, which is a contradiction. $\square$

Proof of Theorem 2. If $G = K_2$, then it is easy to see that G could not be a competition graph of a digraph without loops. If $m(G) > |V(G)|$, we know by Theorem 1 that G is not even the competition graph of a digraph with loops allowed.

To prove the converse, suppose $G \neq K_2$ and $m(G) \leq |V(G)|$. Let $C_1, C_2, \ldots, C_p$ be a collection of cliques covering $E(G)$ and such that $p \leq |V(G)|$.

If $p=0$, then G has no edges and the result is trivial. Suppose $p=1$. Then $C_1 = V(G)$ less isolated vertices and $G = K_m$ plus isolated vertices, where K_m is the complete graph of m vertices. Then it is easy to show that G is a competition graph of a digraph without loops since $G \neq K_2$.

We now consider the case fixed $p > 1$. Let $\alpha = \sum |C_i|$ be the *score* of the clique cover. Then we argue by induction on α, i.e., we prove that if p is a fixed number larger than 1 and at most $|V(G)|$, then if G has an edge covering by p cliques and the score is α, then G is the competition graph of a digraph without loops. The minimum value of α is 0, in which case all $C_i = \emptyset$. Then G has no edges and is clearly a competition graph of a digraph without loops. Suppose the result is true for clique covers of p cliques of score $<\alpha$ and suppose $C_1, C_2, \ldots, C_p$ is a clique cover of score α.

Let $D_i = V(G) - C_i$. We seek a system of distinct representatives (SDR) for $\{D_1, D_2, \ldots, D_p\}$. By Philip Hall's Theorem, such an SDR exists iff for all $k = 1, 2, \ldots, p$,

$$|D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k}| \geq k. \tag{1}$$

If (1) holds for all k, then by Theorem 3, G is a competition graph of a digraph without loops. Suppose then that (1) fails, i.e.,

$$|D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k}| < k$$

for some $D_{i_1}, D_{i_2}, \ldots, D_{i_k}$. If $k=1$, then $D_{i_1} = \emptyset$ and $C_{i_1} = V(G)$, so $G = K_n$. Since $G \neq K_2$, G is clearly a competition graph of a digraph without loops. Now suppose $k \geq 2$ and suppose

$$\{D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k}\} = \{x_1, x_2, \ldots, x_l\},$$

$l < k$. Let $W = V - \{x_1, x_2, \ldots, x_l\}$. Note that $W \subseteq C_{i_j}$ for all j. Note also that since $l < k \leq p \leq |V(G)|$, $W \neq \emptyset$. Now each C_{i_j}, $j = 1, 2, \ldots, k$, is of the form $W \cup T_{i_j}$, where $T_{i_j} \subseteq \{x_1, x_2, \ldots, x_l\}$. Now if no $T_{i_r} \subseteq \bigcup_{j \neq r} T_{i_j}$, then the collection $T_{i_1}, T_{i_2}, \ldots, T_{i_k}$ is a k-formation. Since $k \geq 2$, this is impossible by the lemma. We conclude that for some r, $T_{i_r} \subseteq \bigcup_{j \neq r} T_{i_j}$. Define C_i' to be C_i if $i \neq i_r$ and to be T_{i_r} if $i = i_r$. Note that T_{i_r} is still a clique, being a subset of C_{i_r}. Also, the only edges of G that are in C_{i_r} but not T_{i_r} are edges $\{w, w'\}$ for $w, w' \in W$ or $\{w, x\}$ for $w \in W$, $x \in T_{i_r}$. But each edge $\{w, w'\}$ is contained in all C_{i_j}' for $j \neq r$, and one such C_{i_j}' exists since $k \geq 2$. And each edge $\{w, x\}$ is contained in some C_{i_j}, $j \neq r$, since w is in all C_{i_j} and x is in some C_{i_j}, $j \neq r$. Thus, $C_1', C_2', \ldots, C_p'$ is a collection of $p \leq |V(G)|$ cliques which covers $E(G)$. Moreover, the score $\sum |C_i'|$ of this collection is smaller than the score $\alpha = \sum |C_i|$, since $W \neq \emptyset$ and so $T_{i_r} \subsetneq C_{i_r}$. By inductive assumption, the existence of the cliques $C_1', C_2', \ldots, C_p'$ implies G is a competition graph of a digraph without loops. $\square$

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